Frobenius method and invariants for one-dimensional time-dependent Hamiltonian systems

2001 J. Phys. A: Math. Gen. 34 1005

(http://iopscience.iop.org/0305-4470/34/5/306)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 1005–1017 www.iop.org/Journals/ja PII: S0305-4470(01)17162-3

Frobenius method and invariants for one-dimensional time-dependent Hamiltonian systems

F Haas

Laboratorio Nacional de Computação Científica, LNCC, Av. Getúlio Vargas, 333, 25651-070, Petrópolis, RJ, Brazil

Received 12 September 2000, in final form 4 December 2000

Abstract

We apply the Frobenius integrability theorem in the search for invariants for one-dimensional Hamiltonian systems with a time-dependent potential. We obtain several classes of potential functions for which the Frobenius method ensures the existence of a two-dimensional foliation to which the motion is constrained. In particular, we derive a new infinite class of potentials for which the motion is assuredly restricted to a two-dimensional foliation. In some cases, the Frobenius method allows the explicit construction of an associated invariant. The inverse result is proven that, if an invariant is known, then it can always be furnished by the Frobenius method.

PACS numbers: 1130, 0220, 0230C, 0230E, 0230H, 0545, 4520J

1. Introduction

The search for invariants (also called constants of motion or first integrals) for dynamical systems is a classical topic in nonlinear science, and over the past two decades there has been intensive research on the subject. A comprehensive review of the recent advances on the field can be found in [1]. There are several techniques for the construction of invariants, among which are the direct method [2], application of Noether's theorem [3], utilization of Lie symmetries [4], Painlevé's analysis [5] and the Darboux method [6]. None of these methods has a universal character, and in most cases one or more *ad hoc* assumptions have to be made for obtaining concrete results. For instance, in the case of the Noether and Lie approaches a useful simplification is achieved by considering point symmetries only [7, 8].

In this paper, we focus our attention on Hamiltonian systems with a time-dependent Hamiltonian function of the form

$$H(q, p, t) = \frac{1}{2}p^2 + V(q, t) \tag{1}$$

where $V(q, t)$ is a time-dependent potential function and p and q are coordinates on a two-dimensional phase space. Hamiltonians of type (1) appear in several branches of physics such as plasma physics and quantum mechanics. Here, we are interested in the construction of invariants for suitable classes of potentials. By invariant, we understand precisely a function $I = I(q, p, t)$ globally defined and having the property

$$\dot{I} = \frac{\partial I}{\partial t} + p\frac{\partial I}{\partial q} - \frac{\partial V}{\partial q}\frac{\partial I}{\partial p} = 0 \tag{2}$$

where the dot represents time differentiation. In other words, equation (2) says that the function I is constant along the trajectories of the canonical equations of motion. The existence of an invariant immediately prevents the appearance of chaos, since when a first integral is available the motion is restricted to a two-dimensional surface $I(q, p, t) =$ constant. As is well known, if the independent variable is continuous, chaos may take place only in dynamical systems with three or more dimensions. Note that an explicit time dependence of I does not modify at all the fact that chaos is impossible if an invariant is known to exist (provided $I(q, p, t) =$ constant defines a sufficiently smooth surface).

To begin our analysis, we write the dynamical vector field associated with the canonical equations of motion,

$$\boldsymbol{u} = \frac{\partial}{\partial t} + p\frac{\partial}{\partial q} - \frac{\partial V}{\partial q}\frac{\partial}{\partial p}. \tag{3}$$

Now, the definition (2) reads $\boldsymbol{u}(I) = 0$. The dynamical vector field $\boldsymbol{u}$ is defined in a three-dimensional space, with coordinates q, p and t. Incidentally, some years ago [9, 10] a method was introduced for the derivation of invariants for three-dimensional dynamical systems. This method relies on the Frobenius integrability theorem and has been used in the analysis of the three-dimensional Lotka–Volterra system and the May–Leonard and Lorenz systems [9, 10]. Presently, we apply the Frobenius method in order to derive integrable classes of Hamiltonians of the form (1). The large number of integrable three-dimensional Lotka–Volterra systems detected by the Frobenius method is a good reason for optimism. Indeed, as can be seen in the following, we are able to find several integrable potentials by using the Frobenius method with some auxiliary hypothesis. In particular, we find a whole new infinite family of potentials for which the motion is shown to be restricted to a two-dimensional surface. Finally, we prove the result that, if an invariant is known to exist, then it can be derived by the Frobenius method. This last result shows that the Frobenius approach for invariants for one-dimensional time-dependent Hamiltonian systems necessarily reproduces all known integrable cases in the literature.

This paper is organized as follows. In section 2 we describe the Frobenius method and derive the basic equation to be solved in the case of Hamiltonian systems with a Hamiltonian of the form (1). In section 3, we use a linear ansatz in momentum to obtain particular solutions of the basic equation derived in section 2. As a result, we derive the invariant linear in momentum for the forced time-dependent harmonic oscillator [11] and Sarlet's invariant [12–14]. In section 4 we consider functions which are rational in momentum, and thus obtain an infinite family of potentials amenable to the Frobenius method. As particular cases in this family, we found the potentials having an invariant quadratic in momentum [11] and Giacomini potentials [15, 16]. Also we show a class of 'weakly integrable' potentials, which in general do not have an invariant but for which the Frobenius method works. Section 5 is reserved for conclusions.

2. The Frobenius method and basic equations

The Frobenius theorem deals with the integrability conditions for Pfaff systems of exterior differential equations. More precisely, let $\{\boldsymbol{\theta}^\alpha = 0, \alpha = 1, \ldots, N\}$ be a Pfaff system, where the

$\boldsymbol{\theta}^\alpha$ are 1-forms, defined on a manifold M. This Pfaff system is said to be completely integrable when it is algebraically equivalent to a system of exact differentials. *The Frobenius theorem* [17, p 245] states that the Pfaff system $\{\boldsymbol{\theta}^\alpha = 0, \alpha = 1, \ldots, N\}$ is completely integrable if and only if

$$\mathrm{d}\boldsymbol{\theta}^\alpha \wedge \boldsymbol{\theta}^1 \wedge \ldots \wedge \boldsymbol{\theta}^N = 0 \tag{4}$$

for $\alpha = 1, \ldots, N$. Let $\{\boldsymbol{u}^\alpha, \alpha = 1, \ldots, N\}$ be the set composed by the contravariant vectors associated with the 1-forms $\boldsymbol{\theta}^\alpha$. The integrability criterion (4) means that the Lie bracket of any two contravariant vectors $\boldsymbol{u}^\alpha, \boldsymbol{u}^\beta$ is a linear combination of the elements of $\{\boldsymbol{u}^\alpha, \alpha = 1, \ldots, N\}$.

The Frobenius method as applied to three-dimensional dynamical systems has already been described in [9]. Here, we only show the essentials of the procedure, with a view to application to one-dimensional time-dependent Hamiltonian systems. The basic aim in the Frobenius procedure for the construction of invariants of motion is to find a vector field $\boldsymbol{v}$ given by

$$\boldsymbol{v} = A\frac{\partial}{\partial t} + B\frac{\partial}{\partial q} + C\frac{\partial}{\partial p} \tag{5}$$

which is compatible with the dynamical vector field. In equation (5), $A = A(q, p, t)$, $B = B(q, p, t)$ and $C = C(q, p, t)$ are functions to be determined. Compatibility means that

$$[\boldsymbol{u}, \boldsymbol{v}] = \alpha\boldsymbol{u} + \beta\boldsymbol{v} \tag{6}$$

where [,] represents the Lie bracket, $[\boldsymbol{u}, \boldsymbol{v}] = \boldsymbol{u}\boldsymbol{v} - \boldsymbol{v}\boldsymbol{u}$, and $\alpha = \alpha(q, p, t)$ and $\beta = \beta(q, p, t)$ are functions (not necessarily constants) on phase space and time. Equation (6) implies that the dynamical vector field, the compatible vector field and their Lie bracket are linearly dependent. If $\beta = 0$, the compatibility condition (6) reduces to the statement that $\boldsymbol{v}$ is a generator of Lie symmetries for the dynamical system [4]. For $\beta \neq 0$, there is a generalization.

If a compatible vector field can be found, then the Frobenius theorem ensures the existence of a maximal connected leaf which is tangent to both $\boldsymbol{u}$ and $\boldsymbol{v}$, in each point in the three-dimensional space with coordinates (q, p, t). Moreover, the union of all such leaves provides a foliation, that is, each point (q, p, t) is contained in one and only one leaf. However, this two-dimensional foliation is not necessarily defined in terms of the level surfaces of some function $I(q, p, t)$. Nevertheless, for the derivation of an invariant common to the dynamical and the compatible vector fields we assume the existence of such a function, so that $I(q, p, t) =$ constant provides a representation of the foliation. The fact that $\boldsymbol{u}$ and $\boldsymbol{v}$ are tangent to the maximal connected leaf implies that $\boldsymbol{u}(I) = \boldsymbol{v}(I) = 0$.

Suppose that a compatible vector field $\boldsymbol{v}$ is available. The natural question in this circumstance is how to construct an associated invariant $I(q, p, t)$. The procedure for the derivation of an invariant in the Frobenius method is identical to the construction of first integrals by Lie symmetry methods [4]. By definition, $\boldsymbol{u}(I) = \boldsymbol{v}(I) = 0$, since the maximal surface is tangential to both the dynamical and the compatible vector field. Consider

$$\boldsymbol{v}(I) = 0. \tag{7}$$

Denoting the integral surfaces of the compatible vector field by $f(q, p, t) =$ constant, $g(q, p, t) =$ constant, so that $\boldsymbol{v}(f) = \boldsymbol{v}(g) = 0$, we have that

$$I = I(f, g) \tag{8}$$

is the general solution to (7). As shown in [9], substitution of the form (8) in the condition $\boldsymbol{u}(I) = 0$ leads to a first-order ordinary differential equation,

$$\frac{\mathrm{d}f}{\mathrm{d}g} = \Lambda(f, g) \tag{9}$$

where Λ is a function depending on f and g only. Expressing the constant of integration of the latter equation in terms of $f(q, p, t)$ and $g(q, p, t)$, we obtain the invariant associated with the compatible vector field. As seen by the reasoning in this paragraph, the Lie and the Frobenius strategies for derivation of invariants are indeed equal, once $\boldsymbol{v}$ is known. The sole difference between the two methods is that the compatible vector field is not necessarily a generator of Lie symmetries.

Equation (7) is a first-order linear partial differential equation which has to be solved by the method of characteristics. One may expects, *a priori*, that this equation is easily to handle than the original dynamical system. Otherwise, the existence of a compatible vector field only ensures that the motion is restricted to a two-dimensional foliation, but does not provide the explicit form of an associated invariant. Another possible drawback in the procedure is the resolution of (9). Systems having a compatible vector field but for which one is not able to find the associated constant of motion explicitly were termed weakly integrable in the literature [9]. In section 4, we show an infinity family of potential functions which pertains to such a weakly integrable category.

Turning our attention to our specific problem, the study of one-dimensional time-dependent Hamiltonian systems, let us consider the dynamical and compatible vector fields (3)–(5). Actually, there is no loss of generality in taking $A = 0$ and $B = 1$ in the definition of the compatible vector field. Indeed, suppose that the compatibility condition (6) is fulfilled. Defining a new vector field $\boldsymbol{v}'$ by

$$\boldsymbol{v}' = \mu\boldsymbol{u} + \nu\boldsymbol{v} \tag{10}$$

we obtain, from (6),

$$[\boldsymbol{u}, \boldsymbol{v}'] = \left(\boldsymbol{u}(\mu) + \alpha\nu - \mu\left(\frac{\boldsymbol{u}(\nu)}{\nu} + \beta\right)\right)\boldsymbol{u} + \left(\frac{\boldsymbol{u}(\nu)}{\nu} + \beta\right)\boldsymbol{v}'. \tag{11}$$

Thus, $\boldsymbol{u}$ and $\boldsymbol{v}'$ are compatible. Now, define

$$\mu = \frac{A}{Ap - B} \qquad \nu = \frac{1}{B - Ap}. \tag{12}$$

We always take $B \neq Ap$, so that there is no singularity in the denominator of μ and ν as given in (12). The case $B = Ap$ can be shown to give only trivial results (dynamical and compatible vector fields are linearly dependent). With the choice (12), we have

$$\boldsymbol{v}' = \frac{\partial}{\partial q} + \left(\frac{C + A\partial V/\partial q}{B - Ap}\right)\frac{\partial}{\partial p}. \tag{13}$$

For all compatible vector fields $\boldsymbol{v}$, we can construct an associated compatible vector field $\boldsymbol{v}'$ as defined by (13). In conclusion, vector fields of the form

$$\boldsymbol{v} = \frac{\partial}{\partial q} + C(q, p, t)\frac{\partial}{\partial p} \tag{14}$$

are all that is needed in our calculations. There is no real gain in considering compatible vector fields which are more general than (14). Observe that the use of compatible vector fields like (14) is analogous to the use of generators in evolutionary form, in the theory of Lie extended groups [4]. However, here there is even more simplification, since the coefficient of $\partial/\partial q$ in $\boldsymbol{v}$ was set to unity. Finally, a major advantage of using the reduced form (14) is that the integral surface $g = t =$ constant of the compatible vector field is known in advance. There remains only the task of computing the second integral surface.

Let us study the consequences of the compatibility condition. Computing the Lie bracket of the dynamical vector field and $\boldsymbol{v}$ as given in (14), we find

$$[\boldsymbol{u}, \boldsymbol{v}] = -C\frac{\partial}{\partial q} + \left(\boldsymbol{u}(C) + \frac{\partial^2 V}{\partial q^2}\right)\frac{\partial}{\partial p}. \tag{15}$$

Now using the compatibility condition (6), we determine the coefficients α and β,

$$\alpha = 0 \qquad \beta = -C. \tag{16}$$

In general β is not zero, so that the compatible vector field is not a generator of Lie symmetries.

The compatibility requirement (6) comprises three equations, since we are dealing with vector fields in three dimensions. Two of these equations were used to calculate α and β. The third equation gives

$$\boldsymbol{u}(C) + C^2 = -\frac{\partial^2 V}{\partial q^2}. \tag{17}$$

The latter equation may be used to determine simultaneously the potential and the function C, and is the basic tool in the Frobenius method for one-dimensional Hamiltonian systems with a time-dependent potential. In the next sections, we show several solutions for (17). To deal with (17) we have to suppose appropriate particular forms for the function C. As remarked in the introduction, most methods of derivation of invariants rely on some assumptions. The Frobenius approach is not an exception.

To end this section, we comment on the possible equivalence between the methods of Lie and Frobenius for the construction of invariants. Indeed, if a compatible vector field $\boldsymbol{v}$ is known, one may wonder whether a generator $\boldsymbol{v}'$ of Lie symmetries may be obtained. So, consider $\boldsymbol{v}'$ of the form (10). The Frobenius method generalizes Lie's approach by allowing a non-zero β function in equation (6). A look at equation (11) shows that we have a Lie symmetry if we choose ν so that

$$\boldsymbol{u}(\nu) + \beta\nu = 0. \tag{18}$$

Any vector field $\boldsymbol{v}'$ of the form (10), with arbitrary μ and with ν satisfying (18), is a generator of Lie symmetries provided $\boldsymbol{v}$ is a compatible vector field. However, the difficult point in this reasoning is that (18) is a first-order linear partial differential for ν, which can be solved in general only if the integral surfaces of the dynamical vector field are known. Obviously, *a priori* this information is not available, at least for non-trivial systems.

In sections 3 and 4, we solve the fundamental equation (17) for selected functional forms of $C(q, p, t)$. We use C in the form of functions which are linear and rational in momentum. These two cases are treated separately.

3. $C(q, p, t)$ linear in momentum

As remarked earlier, to obtain definite results some assumption must be made on the functional dependence of $C(q, p, t)$. In this section, we consider

$$C = C_0 + C_1 p \tag{19}$$

where $C_0 = C_0(q, t)$ and $C_1 = C_1(q, t)$ are functions depending only on position and time. Inserting the linear ansatz (19) into (17), it results that a quadratic polynomial in momentum must be identically zero. Equating to zero the coefficients of the various powers of p, we find a system of partial differential equations,

$$\frac{\partial C_1}{\partial q} + C_1^2 = 0 \tag{20}$$

$$\frac{\partial C_1}{\partial t} + \frac{\partial C_0}{\partial q} + 2\, C_0 C_1 = 0 \tag{21}$$

$$\frac{\partial C_0}{\partial t} + C_0^2 - C_1 \frac{\partial V}{\partial q} + \frac{\partial^2 V}{\partial q^2} = 0. \tag{22}$$

This system is nonlinear. To solve it, first observe that (20) admits two types of solution,

$$C_1 = 0 \tag{23}$$

and

$$C_1 = \frac{1}{q - \sigma} \tag{24}$$

where $\sigma = \sigma(t)$ is an arbitrary function. The two branches (23) and (24) will be studied separately.

3.1. Linear invariants

Let us first consider the case $C_1 = 0$. In this circumstance, equation (21) may be readily solved, yielding

$$C_0 = \frac{\dot\rho}{\rho} \tag{25}$$

where $\rho = \rho(t) \neq 0$ is an arbitrary function. The form (25) was chosen for later convenience. We can now solve for the potential using (22), obtaining

$$V = V_0(t) - \frac{\dot F(t) q}{\rho} - \frac{\ddot\rho\, q^2}{2\rho} \tag{26}$$

where $V_0(t)$ and $F(t)$ are new arbitrary functions of time. In fact, we can set $V_0 = 0$ without loss of generality, since it does not contribute to the equations of motion.

The potential (26) corresponds to the forced time-dependent harmonic oscillator system. To construct an invariant for it using the Frobenius method, consider the associated compatible vector field, which, according to (14) and (19), reads

$$\boldsymbol{v} = \frac{\partial}{\partial q} + \frac{\dot\rho}{\rho} \frac{\partial}{\partial p}. \tag{27}$$

The integral surfaces of $\boldsymbol{v}$ are specified by

$$f = p - \frac{\dot\rho\, q}{\rho} \qquad g = t. \tag{28}$$

Computing $\mathrm{d}f/\mathrm{d}t$ along the trajectories of the dynamical system, we obtain

$$\frac{\mathrm{d}f}{\mathrm{d}g} = -\frac{\dot\rho\, f}{\rho} + \frac{\dot F}{\rho}. \tag{29}$$

As $\rho = \rho(g)$, the right-hand side of the last equation is indeed a function of f and g only, in accordance with the general result (9). Solving (29), we obtain the well known [11] invariant linear in momentum for the forced time-dependent harmonic oscillator,

$$I = \rho\, p - \dot\rho\, q - F. \tag{30}$$

3.2. Sarlet's potential

The previous calculations show how the Frobenius method works. Let us consider the less trivial solution (24) for (20) and proceed to the solution of (21) and (22). Taking into account equations (24) and (21), we obtain

$$C_0 = -\frac{\dot{\sigma}(q-\sigma)+2\gamma}{(q-\sigma)^2} \tag{31}$$

where $\gamma = \gamma(t)$ is an arbitrary function of time. With C_0 and C_1 specified by (24) and (31), we can find the potential using (22). It reads

$$V = V_0(t) - \ddot{\sigma}(q-\sigma) - \frac{\ddot{\rho}\, q^2}{2\rho} - \frac{\gamma^2}{2(q-\sigma)^2} - \dot{\gamma}\log(q-\sigma) \tag{32}$$

introducing the new arbitrary functions $V_0 = V_0(t)$ and $\rho = \rho(t) \neq 0$.

Having solved the system (20)–(22), there remains the task of obtaining the associated invariant using the compatible vector field, which is in the present case

$$\boldsymbol{v} = \frac{\partial}{\partial q} + \frac{(q-\sigma)(p-\dot{\sigma})-2\gamma}{(q-\sigma)^2}\frac{\partial}{\partial p}. \tag{33}$$

The associated invariant surfaces are specified by

$$f = \frac{p-\dot{\sigma}}{q-\sigma} - \frac{\gamma}{(q-\sigma)^2} \qquad g = t. \tag{34}$$

Along the canonical equations of motion with potential (32), we obtain

$$\frac{\mathrm{d}f}{\mathrm{d}g} = -f^2 + \frac{\ddot{\rho}}{\rho} \tag{35}$$

i.e. a Riccati equation. Note that the general result (9) is indeed verified, since $\rho = \rho(g)$. The general solution for (35) is

$$f = \frac{\dot{\rho}}{\rho} - \frac{1}{\rho^2\left(I - \int^t \mathrm{d}t'/\rho^2\right)} \tag{36}$$

where I is the constant of integration for the Riccati equation. Actually, I is the invariant of the problem, and, using (34) and (36), we have

$$I = \int^t \frac{\mathrm{d}t'}{\rho^2} - \frac{(q-\sigma)/\rho}{\rho(p-\dot{\sigma}) - \dot{\rho}(q-\sigma) - \gamma\rho/(q-\sigma)}. \tag{37}$$

The potential (32) and the invariant (37) are not new, being derived by the first time by Sarlet and then by a variety of methods [12–14].

In this section, we have shown how the Frobenius procedure works, in the elementary case of functions $C(q, p, t)$ that are linear in momentum. In this way, we have obtained some already known results. In order to find new integrable or weakly integrable potentials, a more complicated momentum dependence of $C(q, p, t)$ must be utilized. It can be easily shown that higher-order polynomial forms of $C(q, p, t)$ do not give anything new. Hence, we proceed differently in the next section, taking a rational form for $C(q, p, t)$.

4. $C(q, p, t)$ rational in momentum

In this section we take the following ansatz for the solution of the basic equation (17):

$$C = \frac{C_0 + C_1 p}{p - C_2} \tag{38}$$

where C_0, C_1 and C_2 are functions of coordinate and time only. We also assume that $C_0 + C_1 C_2 \neq 0$, so that $\partial C/\partial p \neq 0$. Substitution of (38) into (17) yields an equation implying that a cubic polynomial in momentum is identically zero. Again, the coefficient of equal powers in momentum must be zero, resulting in a system of partial differential equations for C_0, C_1, C_2 and the potential. Considering the coefficient of p^3, we find that C_1 is a function of time only. For later convenience, we represents the solution as

$$C_1 = \dot{\rho}/\rho \tag{39}$$

where $\rho = \rho(t)$ is an arbitrary non-zero function of t. Taking into account (39), one can show that the term proportional to p^2 in the basic system of equations yields

$$C_0 = -\ddot{\sigma} - \frac{\ddot{\rho}}{\rho}(q - \sigma) - \frac{\dot{\rho}}{\rho} C_2 - \frac{\partial V}{\partial q} \tag{40}$$

where $\sigma = \sigma(t)$ is a new arbitrary function of time. Equation (40) expresses C_0 in terms of C_2 and V.

The last equations, corresponding to the linear term in momentum and the remaining term, involves C_1 and V. After some algebra, we can transform this system into an equivalent one, consisting of an equation for C_2 only,

$$\frac{\partial C_2}{\partial t} + C_2 \frac{\partial C_2}{\partial q} = \ddot{\sigma} + \frac{\ddot{\rho}}{\rho}(q - \sigma) \tag{41}$$

and an equation determining the potential,

$$\frac{\partial V}{\partial t} + C_2 \frac{\partial V}{\partial q} = -\frac{2\dot{\rho}}{\rho} V + \left(\sigma \dddot{\rho} + (\sigma \dot{\rho} + \rho \dot{\sigma}) \frac{\ddot{\rho}}{\rho} - 2\dot{\rho}\ddot{\sigma} - \rho \dddot{\sigma}\right) \frac{q}{\rho}$$
$$-\left(\frac{\dddot{\rho}}{\rho} + \frac{\dot{\rho}\ddot{\rho}}{\rho^2}\right) \frac{q^2}{2} - \left(\frac{\ddot{\rho}}{\rho}(q - \sigma) + \ddot{\sigma}\right) C_2 + \frac{1}{\rho^2} \frac{\mathrm{d}}{\mathrm{d}t}(\rho^2 V_0(t)) \tag{42}$$

where $V_0(t)$ is an arbitrary function of time.

The strategy for solving (41) and (42) is clear. First, we have to solve (41), obtaining C_2 in terms of (q, t). Then, inserting this solution into (42), we arrive at a well defined partial differential equation for the potential. If its solution is available, we can obtain C_0 and the corresponding associated compatible vector field using (40).

Fortunately, the solution for equation (41) is available [18] for arbitrary ρ and σ, and reads

$$C_2 = \frac{1}{\rho} (\dot{\rho}(q - \sigma) + \rho \dot{\sigma} + Q) \tag{43}$$

where $Q = Q(q, t)$ is implicitly defined according to

$$Q = F\left(\frac{q - \sigma}{\rho} - Q \int^t \frac{\mathrm{d}t'}{\rho^2}\right) \tag{44}$$

where F is an arbitrary function of the indicated argument. As F depends on Q, the solution does indeed have an implicit character. However, the implicit function theorem ensures that we can always solve locally for $Q(q, t)$ under suitable conditions on F. With $Q(q, t)$, we can write C_2 using (43) and then equation (42) for the potential. Note that the appearance of implicit relations is not new in the theory of integrable systems (see, e.g., [15, 18]).

Taking into account the results of the section, we can write the compatible vector field as

$$\boldsymbol{v} = \frac{\partial}{\partial q} + \left(\frac{\dot\rho}{\rho} - \frac{\rho(\partial V/\partial q + \ddot\sigma) + \ddot\rho(q-\sigma)}{\rho(p-\dot\sigma) - \dot\rho(q-\sigma) - Q}\right)\frac{\partial}{\partial p}. \tag{45}$$

To obtain the compatible vector field explicitly, we have to define the function F in (44) and find the potential solving (42). Remembering that the latter is a linear partial differential equation with two independent variables, we conclude that an additional arbitrary function will appear after solving (42) by characteristics. In the following, we illustrate the whole procedure with specific examples.

4.1. *Quadratic invariants*

Let us first consider the case of invariants which are quadratic in momentum. For this class of solutions, we take $F = 0$ in (44). According to (43), we then have

$$C_2 = \dot\sigma + \frac{\dot\rho}{\rho}(q-\sigma). \tag{46}$$

Inserting this form into (42), we obtain the equation determining the potential. The solution is

$$V = V_0(t) + (\sigma\ddot\rho - \rho\ddot\sigma)\frac{q}{\rho} - \frac{\ddot\rho q^2}{2\rho} + \frac{1}{\rho^2}U\left(\frac{q-\sigma}{\rho}\right) \tag{47}$$

where U is an arbitrary function of the indicated argument. This is an example of how the solution of equation (42) by characteristics yields an additional arbitrary function in the complete solution. According to (45), the compatible vector field is

$$\boldsymbol{v} = \frac{\partial}{\partial q} + \left(\frac{\dot\rho}{\rho} - \frac{\mathrm{d}U/\mathrm{d}\bar q}{\rho^2(\rho(p-\dot\sigma) - \dot\rho(q-\sigma))}\right)\frac{\partial}{\partial p} \tag{48}$$

where

$$\bar q = \frac{q-\sigma}{\rho}. \tag{49}$$

Equation (47) indeed defines the class of potentials admitting an invariant quadratic in momentum [11]. To proceed with the Frobenius method for the derivation of invariants, we use a new momentum variable

$$\bar p = \rho(p-\dot\sigma) - \dot\rho(q-\sigma) \tag{50}$$

so that the compatible vector field reads

$$\boldsymbol{v} = \frac{1}{\rho\bar p}\left(\bar p\frac{\partial}{\partial\bar q} - \frac{\mathrm{d}U}{\mathrm{d}\bar q}\frac{\partial}{\partial\bar p}\right). \tag{51}$$

The integral surfaces of $\boldsymbol{v}$ are

$$f = \frac{\bar p^2}{2} + U(\bar q) \qquad g = t. \tag{52}$$

This yields directly the energy-like invariant

$$I = \frac{\bar p^2}{2} + U(\bar q) \tag{53}$$

since, incidentally, $\mathrm{d}f/\mathrm{d}g = 0$ along the trajectories. However, I is the invariant quadratic in momentum derived in [11], expressed in terms of transformed coordinate and momentum. Actually, equations (49) and (50) are an example of a generalized canonical transformation [11]. In the present context, I as given in (53) corresponds to an integral surface of the compatible vector field.

4.2. *Giacomini potentials*

As another particular choice of arbitrary functions, consider

$$\rho = 1 \qquad \sigma = 0 \qquad V_0 = 0. \tag{54}$$

Examining equations (40) and (42), we obtain

$$C_0 = -\frac{\partial V}{\partial q} \qquad C_2 = -\frac{\partial V/\partial t}{\partial V/\partial q}. \tag{55}$$

Also, instead of using expression (43) for C_2, here it is more useful to rewrite (41) as

$$\frac{\partial C_2/\partial t}{\partial C_2/\partial q} = \frac{\partial V/\partial t}{\partial V/\partial q}. \tag{56}$$

The latter result simply states that C_2 and the potential are not functionally independent, $C_2 = C_2(V)$. Thus, the potential satisfies

$$\frac{\partial V}{\partial t} + C_2(V)\frac{\partial V}{\partial q} = 0. \tag{57}$$

According to (45), the compatible vector field can be expressed as

$$\boldsymbol{v} = \frac{\partial V}{\partial q}\left(\frac{\partial}{\partial V} - \frac{1}{p - C_2(V)}\frac{\partial}{\partial p}\right). \tag{58}$$

By inspection, one of the characteristics of $\boldsymbol{v}$ is a function of p and V only,

$$f = f(p, V) \tag{59}$$

where

$$\frac{\partial f}{\partial V} - \frac{1}{p - C_2(V)}\frac{\partial f}{\partial p} = 0. \tag{60}$$

Using the second characteristic $g = t$, we have, by virtue of (55) and (59), (60),

$$\frac{\mathrm{d}f}{\mathrm{d}g} = 0 \tag{61}$$

so that $f(p, V)$ is a first integral.

For general $C_2(V)$, the solution of (60) is not known, so that the Frobenius method only proves weak integrability. However, there is a connection with the work of Giacomini [15], in which a search of invariants depending on the momentum and the potential was made. Giacomini has found exactly equations (57) for the potential and (60) for the invariant. Giacomini [15] as well as Bouquet and Lewis [16] were able to construct explicit solutions for this system of equations by choosing suitable functions $C_2(V)$. Nevertheless, we have proven the interesting result that the Giacomini potentials are always at least weakly integrable, for arbitrary $C_2(V)$, in the sense that they admit a compatible vector field. Also note that Giacomini's solution specified by $\rho = 1$, $\sigma = 0$ and $V_0 = 0$ does not exclude the case of invariants quadratic in momentum, specified by $F = 0$. However, the intersection of the two cases ($\rho = 1, \sigma = 0, V_0 = 0$ and $F = 0$) only yields the trivial result that energy is conserved for a time-independent potential. We do not pursue any further the use of the Frobenius method for Giacomini potentials, since it will not give new results.

4.3. An infinite class of weakly integrable potentials

As a final example, we consider the choice

$$F(s) = s/k \tag{62}$$

for the arbitrary function in equation (44). Here, s denotes an arbitrary argument of F and k is a non-zero numerical constant. The choice (62) allows one to obtain Q globally using (44), and then C_2 by (43). The result is

$$C_2 = \dot{\sigma} + \frac{(q-\sigma)}{\rho}\left(\dot{\rho} + \left(k + \frac{1}{\rho}\int^t \frac{\mathrm{d}t'}{\rho^2}\right)^{-1}\right). \tag{63}$$

It is possible to proceed to the solution of (42) using this function C_2, for arbitrary ρ, σ and V_0. However, the result is awkward and we content ourselves in taking

$$\sigma = 0 \qquad V_0 = 0 \tag{64}$$

in the continuation. Thus, using (42) and (63), we find the following equation for the potential:

$$\frac{\partial V}{\partial t} + \frac{q}{\rho}\left(\dot{\rho} + \frac{1}{\rho}(T+k)^{-1}\right)\frac{\partial V}{\partial q} = -\frac{2\dot{\rho}}{\rho}V - \left(\frac{\ddot{\rho}}{\rho} + \frac{3\dot{\rho}\ddot{\rho}}{\rho^2} + \frac{2\ddot{\rho}}{\rho^3}(T+k)^{-1}\right)\frac{q^2}{2} \tag{65}$$

where

$$T = \int^t \frac{\mathrm{d}t'}{\rho^2}. \tag{66}$$

The solution for (65) is

$$V = \Gamma(t)\bar{q}^2 + \frac{1}{\rho^2}U(\bar{q}) \tag{67}$$

where

$$\Gamma(t) = -\frac{1}{2\rho^2}\int^t \rho^4 \exp\left(2\int^{t'} \frac{\mathrm{d}t''/\rho^2}{T+k}\right)\left(\frac{\ddot{\rho}}{\rho} + \frac{3\dot{\rho}\ddot{\rho}}{\rho^2} + \frac{2\ddot{\rho}}{\rho^3}(T+k)^{-1}\right)\mathrm{d}t' \tag{68}$$

and U is an arbitrary function of

$$\bar{q} = \frac{q}{\rho}\exp\left(-\int^t \frac{\mathrm{d}t'/\rho^2}{T+k}\right). \tag{69}$$

The associated compatible vector field is

$$\boldsymbol{v} = \frac{\partial}{\partial q} + \left(\frac{\dot{\rho}}{\rho} - \frac{\rho\partial V/\partial q + \ddot{\rho}\, q}{\rho\, p - \dot{\rho}\, q - (T+k)^{-1}q/\rho}\right)\frac{\partial}{\partial p} \tag{70}$$

with V given in (67).

To try to find the integral surfaces of $\boldsymbol{v}$, let

$$\bar{p} = \rho\, p - \dot{\rho}\, q - (T+k)^{-1}q/\rho. \tag{71}$$

In terms of $\bar{p}$ and $\bar{q}$ given in (69), we express the characteristic equation associated with $\boldsymbol{v}$ as

$$\bar{p}\frac{\mathrm{d}\bar{p}}{\mathrm{d}\bar{q}} + (T+k)^{-1}\exp\left(\int^t \frac{\mathrm{d}t'/\rho^2}{T+k}\right)\bar{p} + \rho^3\ddot{\rho}\exp\left(2\int^t \frac{\mathrm{d}t'/\rho^2}{T+k}\right)\bar{q} + 2\rho^2\Gamma\bar{q} + \frac{\mathrm{d}U}{\mathrm{d}\bar{q}} = 0. \tag{72}$$

Remembering that t is simply a parameter in this equation (t = constant is one of the integral surfaces of the compatible vector field), we can identify (72) as an Abel equation of second type. Such an Abel equation of second type is not solvable in terms of elementary functions for arbitrary $U(\bar{q})$. Hence, in general the class of potentials (67) is only weakly integrable. Finally, we observe that, if $\rho = 1$, one obtains the Giacomini class of solutions. However, for $\rho \neq 1$ the potential (67) is new.

5. Conclusion

In this work, we have presented the Frobenius method as an attractive strategy for the integrability analysis of one-dimensional time-dependent Hamiltonian systems, which may be cast in the form of three-dimensional dynamical systems. We have shown that the compatible vector fields to be determined can be considered to depend only on a function $C(q, p, t)$. The compatibility condition then leads to the basic equation (17), determining both C and the potential. Starting from some hypothesis on the functional dependence of C, we have been able to derive some already known integrable potentials, namely the potential with an invariant which is linear in momentum, Sarlet's potential, the potential for systems with a quadratic invariant and Giacomini's class of solutions. However, the potential for the system with a quadratic invariant and Giacomini potentials are only particular examples among a richer class, defined by the solutions of (42). These new solutions depend on the arbitrary functions ρ, σ, F and a further arbitrary function arising after resolution of (42), besides the trivial function $V_0(t)$. In the final part of section 4, we have constructed a particular weakly integrable system contained in this family.

The Frobenius method only ensures weak integrability, without providing a first integral in all cases. In this respect, there is a similarity with the Painlevé analysis [5], which only points out when integrability is likely to occur. Indeed, in the Painlevé analysis the construction of an invariant is a separate issue, to be addressed after the singularity structure of the solutions is understood. Also note that, once a compatible vector field is found, the Frobenius procedure for construction of invariants is entirely analogous to Lie's [3, 4] approach. The difference between the two methods is that a compatible vector field is not necessarily a generator of Lie symmetries.

Even if the existence of a compatible vector field does not allows the construction of a first integral, if a first integral is known it is given by the Frobenius method. Indeed, let us make the transformation

$$C = -\frac{\partial J/\partial q}{\partial J/\partial p} \tag{73}$$

in equation (17), where $J = J(q, p, t)$ is an arbitrary function of p, q and t such that $\partial J/\partial p \neq 0$. A little calculation using the form (73) shows that

$$\boldsymbol{u}(C) + C^2 + \frac{\partial^2 V}{\partial q^2} = -\frac{1}{\partial J/\partial p}\boldsymbol{v}\,(\boldsymbol{u}(J)). \tag{74}$$

Hence, if J is an invariant, then the right-hand side of (74) is identically zero and the basic equation (17) is satisfied by the ansatz (73). Moreover, the choice (73) readily implies that $\boldsymbol{v}(J) = 0$, showing that the first integral J is also a first integral of the compatible vector field. This shows that *all* invariants for one-dimensional motion under a time-dependent potential may be furnished by the Frobenius method. In this sense, the Frobenius method has a universal character.

In most situations, no first integral is available *a priori*. In these cases, the crucial point for the effectiveness of the Frobenius approach is a judicious choice of the form of the compatible vector field. Clearly, it is possible that other functional dependences of the coefficient C on the compatible vector field, different from the choices made in this paper, may lead to useful results. Another question that deserves attention is the extension of the Frobenius method and the notion of weak integrability to higher dimensions. To conclude, we have seen that the Frobenius method is a powerful tool in integrability analysis, and we expect that further results can be produced by its systematic application.

Acknowledgments

The author thanks the Laboratoire de Physique des Milieux Ionisés, where this work was carried out, for their hospitality and the Brazilian agency Conselho Nacional de Desenvolvimento Científico e Tecnológico for financial support.

References

[1] Kaushal R S 1998 *Int. J. Theor. Phys.* **37** 1793
[2] Hietarinta J 1987 *Phys. Rep.* **147** 87
[3] Sarlet W and Cantrijn F 1981 *SIAM Rev.* **23** 467
[4] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (New York: Springer)
[5] Ramani A, Grammaticos B and Bountis T 1989 *Phys. Rep.* **180** 159
[6] Cairó L and Llibre D 2000 *J. Phys. A: Math. Gen.* **33** 2395
[7] Haas F and Goedert J 1999 *J. Phys. A: Math. Gen.* **32** 6837
[8] Haas F and Goedert J 2000 *J. Phys. A: Math. Gen.* **33** 4661
[9] Strelcyn J M and Wojciechowski S 1988 *Phys. Lett.* A **133** 207
[10] Grammaticos B, Moulin-Ollagnier J, Ramani A, Strelcyn J M and Wojciechowski S 1990 *Physica* A **163** 683
[11] Lewis H R and Leach P G L 1982 *J. Math. Phys.* **23** 2371
[12] Leach P G L, Lewis H R and Sarlet W 1984 *J. Math. Phys.* **25** 487
[13] Lewis H R and Leach P G L 1985 *Ann. Phys.* **164** 47
[14] Lewis H R, Leach P G L, Bouquet S and Feix M R 1992 *J. Math. Phys.* **33** 591
[15] Giacomini H 1990 *J. Phys. A: Math. Gen.* **23** L865
[16] Bouquet S and Lewis H R 1996 *J. Math. Phys.* **37** 5496
[17] Choquet-Bruhat Y, DeWitt-Morette C and Dillard-Bleick M 1982 *Analysis, Manifolds and Physics* (Amsterdam: North-Holland)
[18] Pereira L G and Goedert J 1992 *J. Math. Phys.* **33** 2682